A Deduction and Demonstration of Taylor's Formula.

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The following method of deducing the formula for the expansion of $f(x+h)$ in terms of ascending powers of h is of interest, because it does not require the assumption of the possibility of the series nor that it should be differentiable.

The determinant

$$\begin{vmatrix} fx\,, & 1, & x\,, & \ldots\ldots, & x^n \\ fa_0, & 1, & a_0, & \ldots\ldots, & a_0^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ fa_n, & 1, & a_n, & \ldots\ldots, & a_n^n \end{vmatrix} \tag{1}$$

vanishes for the $n+1$ values of x, $a_0 \ldots\ldots a_n$. Its first derivative vanishes for n values of x between these values, by Rolle's theorem, fx being a continuous function for the limits prescribed. Its second derivative vanishes for $n-1$ values of x between these values, and so on, until evidently its n^{th} derivative vanishes for some value u, of x, which lies between the greatest and least of the values $a_0 \ldots\ldots a_n$.

Let Fx represent the above determinant, and for brevity write

$$\zeta^{\frac{1}{2}} = \zeta^{\frac{1}{2}}(a_0 \ldots\ldots a_n).$$

We then have

$$Fx = \zeta^{\frac{1}{2}} fx + \phi x, \tag{2}$$

wherein ϕx is a rational integral function of the n^{th} degree.

This being so, we have

$$\begin{aligned} F(x+h) &= \zeta^{\frac{1}{2}} f(x+h) + \phi(x+h) \\ &= \zeta^{\frac{1}{2}} f(x+h) + \phi x + \frac{h^1}{1!}\phi' x + \ldots\ldots + \frac{h^n}{n!}\phi^n x. \end{aligned} \tag{3}$$

Differentiate (2) n times and multiply these equations through respectively by $h^r/r!$, $(r=1\ldots.n)$, whence results

$$hFx=\zeta^{\ddagger}hf'x+h\phi'x,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{h^{n-1}}{(n-1)!}F^{n-1}x=\zeta^{\ddagger}\frac{h^{n-1}}{(n-1)!}f^{n-1}x+\frac{h^{n-1}}{(n-1)!}\phi^{n-1}x,$$

$$\frac{h^n}{n!}F^n u=\zeta^{\ddagger}\frac{h^n}{n!}f^n u+\frac{h^n}{n!}\phi^n u=0.$$

Subtracting these equations from (3), member by member, we obtain

$$f(x+h)-fx-\frac{h^1}{1!}f'x-\ldots.-\frac{h^{n-1}}{(n-1)!}f^{n-1}x-\frac{h^n}{n!}f^n u$$

$$=\frac{1}{\zeta^{\ddagger}}\left[F(x+h)-Fx-\frac{h^1}{1!}F'x-\ldots.-\frac{h^{n-1}}{(n-1)!}F^{n-1}x\right]. \quad (4)$$

Since the a's are arbitrary, we may shift them as we choose, so put $a_0=x$ and $a_n=x+h$, then $Fx=0$, also $F(x+h)=0$, and the second member of (4) becomes

$$-\frac{\dfrac{h^1}{1!}F'x+\ldots.+\dfrac{h^{n-1}}{(n-1)!}F^{n-1}x}{\zeta^{\ddagger}(x,\ a_1\ldots.a_{n-1},\ x+h)},$$

which takes the indeterminate form $0/0$ whenever $a_1,\ldots.,a_{n-1}=x$.

To evaluate the true form of this ratio when $a_1,\ldots.,a_{n-1}=x$, apply to the numerator and denominator the operator

$$\left(\frac{d}{da_1}\right)^1_{a_1=x}\ldots.\left(\frac{d}{da_{n-1}}\right)^{n-1}_{a_{n-1}=x},$$

$\left(\frac{d}{da_r}\right)^r_{a_r=x}$ causes $F^r x$ to vanish $(r=1\ldots.n-1)$, while

$$\left(\frac{d}{da_1}\right)^1_{a_1=x}\ldots.\left(\frac{d}{da_{n-1}}\right)^{n-1}_{a_{n-1}=x}\zeta^{\ddagger}(x,\ a_0,\ldots.,a_{n-1},\ x+h)=(n-1)!!\,h^n.$$

Hence the true value of the ratio is zero and we have

$$f(x+h)=fx+\frac{h^1}{1!}f'x+\ldots.+\frac{h^{n-1}}{(n-1)!}f^{n-1}x+\frac{h^n}{n!}f^n u. \quad (x<u<x+h)$$

The method of determining the ultimate ratio of an indeterminate form can be developed wholly independent of Taylor's formula (Todhunter's Diff. Calculus, p. 124); it seems, therefore, that the above analysis is free from objection.